

# Active polar two-fluid macroscopic dynamics

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**Abstract.** We study the dynamics of systems with a polar dynamic preferred direction. Examples include the pattern-forming growth of bacteria as well as shoals of fish, flocks of birds and migrating insects. Due to the fact that the preferred direction only exists dynamically, but not statically, the macroscopic variable of choice is the macroscopic velocity associated with the motion of the active units, which are typically biological in nature. We derive the macroscopic equations for such a system and discuss novel static, reversible and irreversible cross-couplings connected to a second velocity as a variable. We analyze in detail how the macroscopic behavior of an active system with a polar dynamic preferred direction compares to other systems with two velocities including immiscible liquids and electrically neutral quantum liquids such as superfluid <sup>4</sup>He and <sup>3</sup>He. We critically discuss changes in the normal mode spectrum when comparing uncharged superfluids, immiscible liquids and active system with a polar dynamic preferred direction. We investigate the influence of a macroscopic hand (collective effects of chirality) on the macroscopic behavior of such active media.

## 1 Introduction

The collective dynamics of active systems has attracted increasing attention of the physics community over the last few years. These are (predominantly) biological systems that are driven out of equilibrium, not by externally applied driving fields, but internally —typically by chemical reactions. As long as the internal driving forces are operating, the units of the active system move, and due to this motion an ordered spatial structure is created. Examples are schools of fish or flocks of birds [1–5], pattern-forming growing bacteria (*e.g.* *Proteus mirabilis*) [6–14], biological motors (myosin and actin) [15–20], or general suspensions of active particles. They are all internally driven by chemical reactions, *e.g.* the ATP to ADP consumption, the animals' metabolism, or any other feeding mechanism. If the internal drive stops, *e.g.* due to lack of nutrients, the system becomes passive, with no motion and no remaining localized collective spatial structures, anymore [21].

Here we will describe the cooperative dynamics of such active systems on the macroscopic level by means of hydrodynamic equations. In particular, we focus on cases, where the spatial structure induced by the active units is of the polar type, *i.e.* there is a specific preferred direction in space that allows to discriminate head and tail, or front and rear, or forward and backward within the structure.

Quite recently, we have given a similar description for the case of axial, non-polar spatial order [22].

Orientalional order is, of course, also present through spontaneous breaking of a continuous symmetry in equilibrium systems, like nematic and polar nematic liquid crystals, representing the two different cases, non-polar and polar symmetry mentioned above. In these equilibrium systems, the spatial structure is static in nature. Similarly, static orientational order can be induced in some systems by strong external (magnetic or electric) fields or mechanical stresses. In these cases the spatial structure and its dynamics is described by a second-rank orientational tensor (which can be used to define a director) [23] or a polar vector [24, 25]. These quantities are static in the sense that they do not change under time reversal. In refs. [26, 27] also active systems have been described by these quantities.

Here, we will take a different point of view and describe the active structure as dynamic, created by the motion of active units. The natural candidate for the variable to describe the dynamic structure is then (in the polar case) the velocity of the active entities. Since there is generally also a non-active background, we are left with two different velocities, the non-active one and the active one, which is finite in the active state. Since we will not deal with the transition from the active to the passive state, we do not need to provide the (system specific) link between the feeding process and the active velocity. For an analysis of

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models incorporating such a transition we refer to [28,29]. For a comprehensive discussion of the nonlinear two-fluid hydrodynamics in passive systems, cf. ref. [30].

The active polar media analyzed here should also be contrasted to uncharged superfluids including superfluid  $^4\text{He}$  and the superfluid phases of  $^3\text{He}$ . In this case the additional hydrodynamic variable characteristic of superfluids, the superfluid velocity,  $\mathbf{v}^s$ , is associated with a spontaneously broken continuous symmetry, namely gauge invariance [31–36]. Therefore there is an additional propagating normal mode, second sound, that vanishes in the long wavelength limit, but is associated with a finite velocity. In this paper we will critically compare similarities and differences in the normal mode spectrum of active polar media and superfluids. We find in particular (compare sect. 3 below) that the active systems studied here have only one pair of left-right propagating modes (sound) in the hydrodynamic limit, but show additional non-hydrodynamic propagating modes at higher frequencies.

To explain our model in more detail, we will set up the two-fluid active polar hydrodynamics in sect. 2.1 and discuss the static (sect. 2.2, 2.3) and dynamic aspects, subsequently (sects. 2.4–2.6). In sect. 3 we discuss some simple solutions of our equations including a discussion of the changes in the structure of the normal modes when compared to simple fluids and superfluids. In addition, we will deal with the case of a chiral polar active system (sect. 2.7).

## 2 Nonlinear two-fluid active dynamics

### 2.1 Relevant dynamic variables

As discussed above, we choose the relative velocity  $\mathbf{F}$ , between the active entities and the passive background as the additional variable describing the active motion and active polar spatial structure in such systems. This vector can be decomposed as  $\mathbf{F} = F\mathbf{f}$  with  $|\mathbf{F}| = F$ . The modulus  $F$  is non-zero in an active state and is a measure of how strong the “degree of activity” is. We assume that there are no chemical reactions between the active part and the background. Hence, the masses are conserved and the respective mass densities,  $\rho_1$  and  $\rho_2$  obey conservation laws *separately*, as does the total density  $\rho = \rho_1 + \rho_2$ ,

$$\dot{\rho}_1 + \nabla_i j_i^{(1)} = 0, \quad (1)$$

$$\dot{\rho}_2 + \nabla_i j_i^{(2)} = 0, \quad (2)$$

$$\dot{\rho} + \nabla_i j_i = 0, \quad (3)$$

with the total mass current  $j_i = j_i^{(1)} + j_i^{(2)}$ ; dots denote partial derivatives w.r.t. time.

In the absence of external forces the total momentum is conserved and the total momentum density,  $g_i$ , follows the conservation law

$$\dot{g}_i + \nabla_j \sigma_{ij}^{\text{tot}} = 0, \quad (4)$$

defining the (total) stress tensor  $\sigma_{ij}^{\text{tot}}$ . Defining the mean velocity of the system as  $\mathbf{v} = \mathbf{g}/\rho$  and the (mass) concentration of the active part as  $\phi = \rho_1/\rho$  (hence  $\rho_2/\rho = 1 - \phi$ ) we have the following relations [30]:

$$\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2 \equiv \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2, \quad (5)$$

$$\mathbf{v}_1 = \rho^{-1} \mathbf{g} + (1 - \phi) \mathbf{F}, \quad (6)$$

$$\mathbf{v}_2 = \rho^{-1} \mathbf{g} - \phi \mathbf{F}, \quad (7)$$

where the relative velocity  $\mathbf{F}$  is related to the momentum densities and the velocities of the active and background part, respectively, by  $\mathbf{F} \equiv \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{g}_1/\rho_1 - \mathbf{g}_2/\rho_2$ . Obviously,  $\mathbf{F}$  transforms as a velocity, *i.e.* it is a polar vector (odd under spatial inversion) that changes sign under time reversal. In ref. [22] the variable  $\mathbf{W}$  that describes a dynamic *axial* spatial structure also is odd under time reversal, but even under spatial inversion. There is no conservation law connected with  $\mathbf{F}$ , since relative velocities give rise to friction and, hence, to dissipation for non-superfluids.

The kinetic energy density connected with the two velocities and densities can be written as

$$\epsilon_{\text{kin}} = \frac{1}{2} \rho_1 \mathbf{v}_1^2 + \frac{1}{2} \rho_2 \mathbf{v}_2^2 = \frac{1}{2} \rho \mathbf{g}^2 + \frac{1}{2} \alpha \mathbf{F}^2, \quad (8)$$

with  $\alpha = \rho\phi(1 - \phi)$ . Since  $\mathbf{F}^2 = F^2$  only contains the modulus, the orientation of  $\mathbf{f}$  is energetically not fixed. Thus, the active polar structure breaks rotational symmetry, spontaneously. This also implies that rotations of  $\mathbf{f}$  are the appropriate hydrodynamic variables. This is in complete analogy to the passive case [24]. In the passive case eq. (8) has the equilibrium ground state  $\mathbf{g} = 0$  and  $F = 0$ . However, the active state is not in equilibrium and as ground state a stationary one, characterized by a finite  $F_s > 0$  (and  $\mathbf{g}_s = 0$ ) and arbitrary  $\mathbf{f}$ , can be used. To accommodate this ground state, a fictitious “active field”  $\mathbf{F}^{\text{act}} = \alpha F_s \mathbf{f}$  can be introduced giving rise to an energy contribution

$$\epsilon_{\text{act}} = -\mathbf{F}^{\text{act}} \cdot \mathbf{F} = -\alpha F_s F. \quad (9)$$

Generally, there can be multiple stationary states, and  $F_s$  might depend on space, but we will not consider these cases.

Finally, since the system is a thermodynamic one, we use the entropy density  $\sigma$  (with the temperature,  $T$ , its conjugate) to describe thermal effects. The energy density  $\epsilon$  is then not independent, but given by the thermodynamic Gibbs relation

$$d\epsilon - T d\sigma = \Pi d\phi + \mu d\rho + \mathbf{v} \cdot d\mathbf{g} + \mathbf{m}' \cdot d\mathbf{F} + \Psi_{ij} d\nabla_j F_i + h'_i d f_i \quad (10)$$

that relates infinitesimal changes of the variables with those of the energy density. We have introduced the osmotic pressure  $\Pi$  and the conjugate to the relative velocity,  $\mathbf{m}'$ . The conjugate quantity  $\mathbf{m}' = \alpha(\mathbf{F} - F_s \mathbf{f})$  is proportional and parallel to the relative velocity and shares

the transformation behavior of the latter. Decomposing  $\mathbf{m}' = m' \mathbf{f}$ , it becomes obvious that  $\mathbf{m}' \cdot d\mathbf{F} = m' dF$  (due to  $\mathbf{f} \cdot d\mathbf{f} = 0$ ), with  $m'$  describing the stiffness w.r.t. changes of the degree of activity  $F$ .

In the case of a static polar structure, an external electric field linearly orients this structure. In the present case of a dynamic ordering, such an external field,  $\mathbf{F}^{\text{ext}}$  has to be a velocity-type quantity (polar vector odd under time reversal), *e.g.* a flow field that acts only on the active entities. If such a physical external field exists, an additional term  $\sim \mathbf{F}^{\text{ext}} \cdot d\mathbf{F}$  will appear in the Gibbs relation.

In the Gibbs relation eq. (10) we have added gradients of the relative velocity,  $d\nabla_j F_i$ , that describe deformations of the polar structure as well as inhomogeneous distributions of the degree of activity. Since the latter is an intrinsic property in an active system, inhomogeneities might be present, even in a bulk system. An example would be a finite gradient in the concentration of the nutrient that fosters active behavior. The conjugate quantity  $\Psi_{ij}$  has to be a second rank tensor that is even under spatial inversion and odd under time reversal, such that the energy is even under inversion and time reversal. Finally, we have introduced for completeness contributions to energy changes due to local rotations of the polar structure,  $df_i$ , that arise because of the anisotropy (the  $f_i$  dependence) of the material tensors. The so-called molecular field  $h'_i$  has the symmetry properties of a velocity and is transverse with  $h'_i f_i = 0$ .

In order to guarantee the energy to be a scalar quantity,  $d\epsilon$  has to vanish for any coordinate rotation. Describing the latter by a constant antisymmetric matrix  $\Omega_{ij} = -\Omega_{ji}$ , there is  $dF_i = \Omega_{ij} F_j$  and  $d\nabla_j F_i = \Omega_{jk} \nabla_k F_i + \Omega_{ik} \nabla_j F_k$  resulting in a compatibility condition for the conjugate quantities [37]

$$h'_i f_j + \Psi_{ki} \nabla_j F_k + \Psi_{ik} \nabla_k F_j = h'_j f_i + \Psi_{kj} \nabla_i F_k + \Psi_{jk} \nabla_k F_i. \quad (11)$$

There are no contributions from the momenta and velocities, since  $\mathbf{g} \parallel \mathbf{v}$  and  $\mathbf{F} \parallel \mathbf{m}'$ .

Since the modulus of  $\mathbf{F}$  is a non-conserved variable and its orientation is the hydrodynamic symmetry variable, it makes sense to decompose it in the Gibbs relation. Neglecting surface effects (*i.e.* allowing for partial integration in the volume integral of the total energy  $E \equiv \int \epsilon dV$ ) one can put the Gibbs relation into a form suitable to derive bulk hydrodynamic equations

$$d\epsilon = T d\sigma + \Pi d\phi + \mu d\rho + \mathbf{v} \cdot d\mathbf{g} + m dF + h_i df_i. \quad (12)$$

The explicit forms of  $m$  and  $h_i$  are given in eqs. (29) and (30), below. Note that  $m$  is a time-reversal symmetric, scalar quantity, while  $h_i$  transforms like a velocity and is perpendicular to  $f_i$ . The conjugate quantities are obtained by partial derivation of the total energy density, which is discussed in the subsequent section. The full expressions of the conjugate quantities are listed in sect. 2.3.

Finally, we will provide an expression for the thermodynamic pressure,  $p$ , that enters the hydrodynamic stress tensor. From the definition  $p = -\partial E / \partial V$  it is clear that the pressure contains all extensive quantities, like mass,

momentum, entropy, and energy, but not the intensive ones, like concentration or relative velocity [30]

$$p = -\epsilon + T\sigma + \rho\mu + \rho^{-1} \mathbf{g}^2 \quad (13)$$

and its gradient can be written as (Gibbs-Duhem equation)

$$\begin{aligned} \nabla_i p = & \sigma \nabla_i T + \rho \nabla_i \mu + g_j \nabla_i v_j - \Pi \nabla_i \phi \\ & - m' \nabla_i F - \Psi_{kj} \nabla_i \nabla_j F_k - h'_j \nabla_i f_j. \end{aligned} \quad (14)$$

Neglecting surface effects in the Navier-Stokes equation, the three last contributions can be simplified to  $-m \nabla_i F - h_j \nabla_i f_j$ .

## 2.2 Statics

In a hydrodynamic description statics is the phenomenological relation between the thermodynamic conjugates and the variables of the system. What is left in the present case is the connection between the scalar quantities  $T, \mu, \Pi$  and the (“molecular”) field  $h_i$  on the one hand, with the state variables  $\sigma, \rho, \phi$  and with  $\nabla_j F_i$  on the other. This is most easily done by adding three contributions to the energy, the gradient energy,  $\epsilon_{\text{grad}}$ , the state energy,  $\epsilon_{\text{state}}$ , and the mixed one,  $\epsilon_{\text{mix}}$ . Standard symmetry considerations [37] lead to

$$\epsilon_{\text{grad}} = \frac{1}{2} K_{ijkl} (\nabla_j F_i) (\nabla_l F_k), \quad (15)$$

$$\begin{aligned} \epsilon_{\text{state}} = & \frac{1}{2} c_{\rho\rho} (\delta\rho)^2 + \frac{1}{2} c_{\sigma\sigma} (\delta\sigma)^2 + \frac{1}{2} c_{\phi\phi} (\delta\phi)(\delta\phi) \\ & + c_{\rho\phi} (\delta\rho)(\delta\phi) + c_{\rho\sigma} (\delta\rho)(\delta\sigma) + c_{\sigma\phi} (\delta\sigma)(\delta\phi), \end{aligned} \quad (16)$$

$$\epsilon_{\text{mix}} = (\sigma_{ijk}^{\sigma} \nabla_k \sigma + \sigma_{ijk}^{\rho} \nabla_k \rho + \sigma_{ijk}^{\phi} \nabla_k \phi) (\nabla_i F_j). \quad (17)$$

This energy has been written in terms of deviations from the stationary state with  $\delta\rho \equiv \rho - \rho_s$ ,  $\delta\sigma \equiv \sigma - \sigma_s$ , and  $\delta\phi \equiv \phi - \phi_s$ , where the stationary values  $\rho_s$ ,  $\sigma_s$ , and  $\phi_s$  are assumed to be constant.

The generalized Frank tensor  $K_{ijkl}$  has to contain an even number of  $f_i$  factors due to time reversal and spatial inversion symmetry, while the mixed tensors  $\sigma_{ijk}^f$  contain an odd number for the same reason. In particular we find

$$\begin{aligned} K_{ijkl} = & \frac{1}{2} K_1 (\delta_{ij}^{\perp} \delta_{kl}^{\perp} + \delta_{il}^{\perp} \delta_{jk}^{\perp}) + K_2 f_p \epsilon_{pij} f_q \epsilon_{qkl} \\ & + K_3 f_l f_j \delta_{ik}^{\perp} + K_4 f_i f_j f_k f_l + K_5 f_i f_k \delta_{jl}^{\perp} \\ & + \frac{1}{4} K_6 (f_i f_l \delta_{kj}^{\perp} + f_j f_k \delta_{il}^{\perp} + f_i f_j \delta_{kl}^{\perp} + f_k f_l \delta_{ij}^{\perp}), \end{aligned} \quad (18)$$

with  $\delta_{ij}^{\perp} = \delta_{ij} - f_i f_j$ . It has the same structure as in the nematic and static polar case, as well as in the axial dynamic one. The first three terms are the proper Frank ones,  $K_4$  and  $K_5$  relate  $\nabla_i F$  with each other, while  $K_6$  describes couplings between  $\nabla_i F$  and  $\nabla_j f_i$ . The energy

$\epsilon_{\text{st}}$  contains the standard thermodynamic susceptibilities, like compressibility, specific heat, thermal expansion, etc. The mixed gradient terms have the same structure as in the dynamic axial case [22]

$$\sigma_{ijk}^f = \sigma_1^x f_i f_j f_k + \sigma_2^x f_j \delta_{ik}^\perp + \sigma_3^x (f_i \delta_{jk}^\perp + f_k \delta_{ij}^\perp), \quad (19)$$

where  $x$  can be either  $\rho$ ,  $\sigma$  or  $\phi$ .

In contrast to the dynamic axial case, there is no static cross-coupling between the momentum and order,  $\sim g_i \nabla_j F_k$ , due to the polar nature of  $\mathbf{F}$ . Because of the time reversal properties of  $\mathbf{F}$ , there is no linear gradient term of the form  $\nabla_i F_i$  possible in the free energy. Such a term is present in the passive polar case and responsible for the possibility of inhomogeneous splay phases [38]. In addition, in the dynamic polar case, there are no couplings between  $\delta F$ ,  $f_i \nabla_i F$ , and  $\text{div} \mathbf{f}$  with the state variables, as is the case in the static polar case [24].

An external electric ( $\mathbf{E}$ ) or magnetic ( $\mathbf{H}$ ) field interacts with the dynamic polar structure in the same way as with a nematic order, *i.e.* through dielectric and diamagnetic anisotropy and flexo- and order-electricity, respectively

$$\epsilon_f = \frac{1}{2} \chi_a^E (\mathbf{F} \cdot \mathbf{E})^2 + \frac{1}{2} \chi_a^H (\mathbf{F} \cdot \mathbf{H})^2 + \zeta_{ijk} E_i \nabla_j F_k, \quad (20)$$

with  $\zeta_{ijk} = \zeta_1 f_i f_j f_k + \zeta_2 f_k \delta_{ij}^\perp + \zeta_3 (f_i \delta_{jk}^\perp + f_j \delta_{ik}^\perp)$ . An external flow field that acts only on the active entities, if it exists, will add a term,  $-\chi_{\text{ext}} \mathbf{F}^{\text{ext}} \cdot \mathbf{F}$ , to  $\epsilon_f$ .

Finally, the total energy density reads

$$\epsilon = \epsilon_{\text{kin}} + \epsilon_{\text{act}} + \epsilon_{\text{grad}} + \epsilon_{\text{state}} + \epsilon_{\text{mix}} + \epsilon_f. \quad (21)$$

It has been set up as a sum of bilinear expressions in the variables. Nevertheless, the resulting state equations are highly nonlinear, because i) the preferred direction  $f_i$  in the material tensors is a dynamic variable, ii) all material coefficients are a function of the scalar state variables,  $\rho$ ,  $\sigma$ , and  $\phi$ , and iii) the kinetic energy is manifestly nonlinear.

### 2.3 Equations of state

According to their definition in the Gibbs relation eqs. (10) or (12), the thermodynamic quantities are obtained by partial derivation of the total energy, *e.g.* by  $\Psi_{ij} = \partial \epsilon / \partial \nabla_j F_i$ . Again, we assume that the values of the conjugate quantities in the stationary state ( $T_s$ ,  $\mu_s$ , and  $\Pi_s$ ) are constant. We only need to consider the deviations from the stationary state (denoted by “ $\delta$ ”, when necessary)

$$\begin{aligned} \Psi_{ij} &= \left( \frac{\partial \epsilon}{\partial \nabla_j F_i} \right)_{\dots} \\ &= K_{ijkl} \nabla_k F_l + \sigma_{ijk}^\rho \nabla_k \rho + \sigma_{ijk}^\sigma \nabla_k \sigma + \sigma_{ijk}^\phi \nabla_k \phi, \end{aligned} \quad (22)$$

$$\begin{aligned} \delta T &= \left( \frac{\partial \epsilon}{\partial \sigma} \right)_{\dots} - \left( \frac{\partial \epsilon}{\partial \sigma} \right)_{|s \dots} \\ &= c_{\sigma\sigma} \delta \sigma + c_{\rho\sigma} \delta \rho + c_{\sigma\phi} \delta \phi - \sigma_{kji}^\sigma \nabla_i \nabla_k F_j, \end{aligned} \quad (23)$$

$$\begin{aligned} \delta \mu &= \left( \frac{\partial \epsilon}{\partial \rho} \right)_{\dots} - \left( \frac{\partial \epsilon}{\partial \rho} \right)_{|s \dots} \\ &= c_{\rho\rho} \delta \rho + c_{\rho\phi} \delta \phi + c_{\rho\sigma} \delta \sigma - \sigma_{kji}^\rho \nabla_i \nabla_k F_j - \frac{1}{2} \mathbf{v}^2 \\ &\quad + \phi(1-\phi)F \left( \frac{1}{2} F - F_s \right) + \frac{1}{2} \phi_s(1-\phi_s)F_s^2, \end{aligned} \quad (24)$$

$$\begin{aligned} \delta \Pi &= \left( \frac{\partial \epsilon}{\partial \phi} \right)_{\dots} - \left( \frac{\partial \epsilon}{\partial \phi} \right)_{|s \dots} \\ &= c_{\phi\phi} \delta \phi + c_{\phi\sigma} \delta \sigma + c_{\phi\rho} \delta \rho - \sigma_{kji}^\phi \nabla_i \nabla_k F_j \\ &\quad + \rho(1-2\phi)F \left( \frac{1}{2} F - F_s \right) + \frac{1}{2} \rho_s(1-2\phi_s)F_s^2, \end{aligned} \quad (25)$$

where the dots mean that all other variables (out of  $\{\sigma, \phi, \rho, g_i, F_i, \nabla_j F_i\}$ ) are held fixed, and the subscript “ $s$ ” denotes the value in the stationary state of that quantity. In the expressions containing  $\sigma_{kji}^x$  we have integrated by parts.

There are specific dynamic-active contributions to the state equations in the last lines of eqs. (24) and (25), which are quadratic in  $F$ . They arise from the state variable dependence of  $\alpha$  in the kinetic energy eq. (8). There is no such contribution in the temperature. These terms are nonlinear, but come without a (possibly small) phenomenological parameter and are only possible in the case of active and dynamic order. There are comparable contributions in the axial dynamic case [22].

When deriving  $m_i$ , the conjugate to  $F_i$ , one has to take into account that the material tensors are written for convenience in terms of  $f_i$  rather than  $F_i$ . In particular we get (suppressing external fields for the moment)

$$m_i = \left( \frac{\partial \epsilon}{\partial F_i} \right)_{\dots} = \alpha(F_i - f_i F_s) + F^{-1} \delta_{ij}^\perp h'_j, \quad (26)$$

with

$$\begin{aligned} h'_i &\equiv + \left( \frac{\partial \epsilon}{\partial f_i} \right)_{F_i, \dots} = \frac{\partial K_{qjkl}}{\partial f_i} (\nabla_j F_q) (\nabla_l F_k) \\ &\quad + \sum_{x=\rho, \sigma, \phi} \frac{\partial \sigma_{qjk}^x}{\partial f_i} (\nabla_k x) (\nabla_q F_j) \end{aligned} \quad (27)$$

Decomposing  $F_i = F f_i$  the Gibbs relation eq. (10) takes the form

$$\begin{aligned} d\epsilon &= T d\sigma + \Pi d\phi + \mu d\rho + \mathbf{v} \cdot d\mathbf{g} + f_i \Psi_{ij} d\nabla_j F \\ &\quad + (m' + \Psi_{ij} \nabla_j f_i) dF + F \Psi_{ij} d\nabla_j f_i \\ &\quad + \delta_{ik}^\perp (h'_k + \Psi_{kj} \nabla_j F) d f_i, \end{aligned} \quad (28)$$

where  $m_i$  and  $\Psi_{ij}$  follow from eqs. (26) and (22), respectively.

Integration by parts leads to eq. (12) with the conjugates

$$m = \left( \frac{\partial \epsilon}{\partial F} \right)_{f_i, \dots} = \alpha(F - F_s) - f_i \nabla_j \Psi_{ij}, \quad (29)$$

$$h_i = \left( \frac{\partial \epsilon}{\partial f_i} \right)_{F, \dots} = \delta_{ik}^\perp (h'_k - F \nabla_j \Psi_{kj}), \quad (30)$$

that contain the longitudinal and transverse parts of the vectors  $h'_i$  and  $\nabla_j \Psi_{ij}$ , respectively; the dots contain  $\{\sigma, \phi, \rho, g_i\}$ .

If electric and magnetic fields are present, the conjugates acquire additional contributions according to the field energy  $\epsilon_f$

$$m \rightarrow m + \chi_a^E F(\mathbf{E} \cdot \mathbf{f})^2 + \chi_a^M F(\mathbf{M} \cdot \mathbf{f})^2, \quad (31)$$

$$h_i \rightarrow h_i + \delta_{ik}^{\perp} f_j F(\chi_a^E E_j E_k + \chi_a^M M_j M_k), \quad (32)$$

and in addition

$$h'_i \rightarrow h'_i + \frac{\partial \zeta_{kjq}}{\partial f_i} E_q \nabla_j F_k, \quad (33)$$

$$\Psi_{ij} \rightarrow \Psi_{ij} + \zeta_{kji} E_k. \quad (34)$$

Of course, the external field not only influences the stiffness of the polar structure, but also  $m$ , the stiffness of fluctuations of the degree of activity.

## 2.4 Dynamic equations

Dynamic equations come in two classes, either conservation laws for the conserved quantities or balance equations for the non-conserved ones. These equations contain the currents and quasi-currents, respectively, which are the sum of a reversible and an irreversible part. This distinction refers to their time reversal behavior and is linked to the entropy production: the reversible parts are non-dissipative with zero entropy production, while the irreversible ones are dissipative with positive entropy production. The latter can be derived from a Lyapunov functional, the dissipation function  $R$ , which is closely related to the entropy production  $R/T$ . Within linear irreversible thermodynamics  $R$  is a bilinear function of the thermodynamic forces, which are generally the gradients of the conjugate quantities discussed in the preceding section. The phenomenological coefficients of that function are the transport parameters of the system.

The reversible parts of the currents do not follow from a Lyapunov functional, but either follow from general symmetry, invariance, or thermodynamic principles or are derived phenomenologically from the zero entropy production requirement. The latter come with phenomenological parameters and a well-known example is the flow alignment coefficient in ordinary nematic liquid crystal. There is a considerable number of such reversible transport parameters in magnetic nematic materials [39,40] and dynamically ordered axial systems [22]. The symmetry-related parts are generally nonlinear, but do not come with a (possibly small) phenomenological parameter. Examples are the isotropic pressure contribution in the stress tensor, the advective transport contributions to the dynamics that describe, in the Eulerian picture, temporal changes of a quantity at a fixed point in space due to material flow (transport contributions), and the convective parts of the time derivative describing the rotational behavior of vectors and tensors (involving the local vorticity).

In a two-fluid description it is *a priori* not fixed, which one (or which combination) of the two velocities present should be chosen in the transport terms. There are phenomenological reversible currents of the form of transport terms that influence the effective advective velocity [30]. To fix the transport term, special (mostly unproven) models have to be employed. On the other hand, one is not completely free in choosing the advective velocity or the convective part, and a certain choice might have consequences for other properties [30]. One restriction is the zero entropy production requirement of such terms.

Here we will start with a consistent ansatz, where all quantities are advected by the mean velocity  $\mathbf{v}$  and are convected by  $\text{curl } \mathbf{v}$ . In the reversible dynamics section we show how phenomenological contributions in the currents might lead to a different effective transport velocity (and convection vorticity) for the particular variables. This procedure is also compatible with our choice of the total mass current being identical to the total momentum density,  $\mathbf{j} = \mathbf{g} = \rho \mathbf{v}$ , which has the further consequence that gradients of the chemical potential  $\nabla_i \mu$  cannot be the source of phenomenological contributions to any current. Under these provisos the dynamic equations read

$$\dot{\rho} + \nabla_i g_i = 0, \quad (35)$$

$$\dot{\sigma} + \nabla_i (\sigma v_i) + \nabla_i j_i^{\sigma} = \frac{R}{T}, \quad (36)$$

$$\dot{\phi} + v_i \nabla_i \phi + \rho^{-1} \nabla_i j_i^{\phi} = 0, \quad (37)$$

$$\dot{g}_i + \nabla_j (v_j g_i + \delta_{ij} p + \sigma_{ij}^{th} + \sigma_{ij}) = 0, \quad (38)$$

$$\dot{F}_i + v_j \nabla_j F_i - F_j \omega_{ij} + X_i = 0, \quad (39)$$

where the latter can be decomposed

$$\dot{F} + v_i \nabla_i F + X = 0, \quad (40)$$

$$\dot{f}_i + v_j \nabla_j f_i - f_j \omega_{ij} + Y_i = 0, \quad (41)$$

with  $X = f_i X_i$  and  $Y_i = F^{-1} \delta_{ik}^{\perp} X_k$ . In the stress tensor there is an additional non-phenomenological, thermodynamic term

$$\sigma_{ij}^{th} = +\frac{1}{2} (\Psi_{jk} \nabla_k F_i + \Psi_{ik} \nabla_k F_j), \quad (42)$$

whose structure resembles the Ericksen stress in ordinary nematics and which has been symmetrized [37] making use of eq. (11). To achieve this goal it is crucial that the convective part of the  $\mathbf{F}$  dynamics is expressed by a vorticity based on the mean velocity,  $\omega_{ij} = (1/2)(\nabla_j v_i - \nabla_i v_j)$ . As will be shown below, however, the effective relevant vorticity may be different due to some phenomenological parts of  $X_i$ .

The dissipative and reversible parts (superscripts  $D$  and  $R$ , respectively) of the phenomenological currents defined in eqs. (36)–(41),  $\{j_i^{\sigma}, j_i^{\phi}, \sigma_{ij}, X_i\}$  (or rather  $X, Y_i$ ) will be discussed subsequently.

## 2.5 Dissipative dynamics

For dissipative processes the entropy production  $\int R/TdV$  has to be positive. Using eq. (10), this requires

$$R = -j_i^{\sigma,D} \nabla_i T - j_i^{\phi,D} \nabla_i \Pi' - \sigma_{ij}^D A_{ij} + Y_i^D h_i + X^D m > 0, \quad (43)$$

with  $\Pi = \rho \Pi'$  and the symmetrized velocity gradient  $A_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ . Note that  $\Omega_{ij} \equiv \frac{1}{2}(\nabla_i v_j - \nabla_j v_i)$  does not enter eq. (43), since rigid rotations must not produce entropy.

Equation (43) also reveals the stationary state conditions

$$\nabla_i T = \nabla_i \Pi' = A_{ij} = h_i = m = 0, \quad (44)$$

where the latter conditions mean  $f_i = \text{const.}$  The dissipation function  $R$  is a time reversal-symmetric scalar and can be written as

$$\begin{aligned} 2R = & \kappa_{ij}(\nabla_i T)(\nabla_j T) + D_{ij}(\nabla_i \Pi')(\nabla_j \Pi') \\ & + 2D_{ij}^T(\nabla_i \Pi')(\nabla_j T) + \nu_{ijkl} A_{ij} A_{kl} \\ & + \mu_{ijk}^T A_{ij} \nabla_k T + \mu_{ijk}^{\Pi} A_{ij} \nabla_k \Pi' \\ & + b^D \delta_{ij}^{\perp} h_i h_j + \xi' m^2 + O(\nabla \mathbf{m}, \nabla \mathbf{h}), \end{aligned} \quad (45)$$

including heat conduction, diffusion and thermodiffusion material tensors ( $\kappa_{ij}$ ,  $D_{ij}$ ,  $D_{ij}^T$ , respectively), which are all of the uniaxial form

$$\pi_{ij} = \pi_{\perp} \delta_{ij}^{\perp} + \pi_{\parallel} f_i f_j \quad (46)$$

for  $\pi \in \{\kappa, D, D^T\}$ . The cross-coupling between flow and temperature and osmotic pressure gradients are specific for dynamic polar systems and the transport tensors  $\mu_{ijk}^x$  ( $x \in \{T, \Pi\}$ ) are of the same form as the  $\sigma_{ijk}^x$  in eq. (19). For dynamic axial structures, such couplings are only possible in the chiral case.

The viscosity related to gradients of the mean velocity is expressed by the tensor  $\nu_{ijkl}$ , which is of the conventional uniaxial (nematic) structure with 5 different viscosity coefficients [41]. Orientational relaxation of the polar structure is described by the transport coefficient  $b^D$ , while the friction due to the existence of a relative velocity (or in other words, the relaxation of the degree of activity,  $F$ ) is characterized by  $\xi'$ . The two latter processes are both governed by a scalar quantity (rather than a tensor), since structure rotations are strictly transverse and the relative flow (is by definition) strictly longitudinal.

Although there are already dissipative processes related to  $h_i$  and  $m$ , we will list here additional viscosity-like dissipative effects that are characteristic for 2-fluid systems

$$\begin{aligned} O(\nabla \mathbf{m}, \nabla \mathbf{h}) = & \nu_{ij}^{FF}(\nabla_i m)(\nabla_j m) + \nu_{ijkl}^{ff}(\nabla_j h_i)(\nabla_l h_k) \\ & + \nu_{ijk}^{Ff}(\nabla_i m)(\nabla_j h_k) + \nu_{ijk}^{F\Pi}(\nabla_i m)A_{jk} \\ & + \nu_{ijk}^{\sigma f}(\nabla_i T)(\nabla_j h_k) + \nu_{ijk}^{\phi f}(\nabla_i \Pi')(\nabla_j h_k) \\ & + \nu_{ijkl}^f(\nabla_j h_i)A_{kl}, \end{aligned} \quad (47)$$

with

$$\nu_{ij}^{FF} = \nu_{\perp}^{FF} f_i f_j + \nu_{\perp}^{FF} \delta_{ij}^{\perp}, \quad (48)$$

$$\begin{aligned} \nu_{ijkl}^{ff} = & \frac{1}{2} \nu_1^{ff} (\delta_{ij}^{\perp} \delta_{kl}^{\perp} + \delta_{il}^{\perp} \delta_{jk}^{\perp}) + \nu_2^{ff} f_p \epsilon_{pij} f_q \epsilon_{qkl} \\ & + \nu_3^{ff} f_l f_j \delta_{ik}^{\perp}, \end{aligned} \quad (49)$$

$$\nu_{ijk}^{Ff} = \nu^{Ff} (f_i \delta_{jk}^{\perp} + f_j \delta_{ik}^{\perp}), \quad (50)$$

$$\nu_{ijk}^{\sigma f} = \nu^{\sigma f} (f_i \delta_{jk}^{\perp} + f_j \delta_{ik}^{\perp}), \quad (51)$$

$$\nu_{ijk}^{\phi f} = \nu^{\phi f} (f_i \delta_{jk}^{\perp} + f_j \delta_{ik}^{\perp}), \quad (52)$$

$$\nu_{ijk}^F = \nu_1^F f_i f_j f_k + \nu_2^F f_i \delta_{jk}^{\perp} + \frac{1}{2} \nu_3^F (f_j \delta_{ik}^{\perp} + f_k \delta_{ij}^{\perp}), \quad (53)$$

$$\begin{aligned} \nu_{ijkl}^f = & \nu_1^f \delta_{ij}^{\perp} f_k f_l + \frac{1}{2} \nu_2^f f_j (f_k \delta_{il}^{\perp} + f_l \delta_{ik}^{\perp}) + \nu_3^f \delta_{ij}^{\perp} \delta_{kl}^{\perp} \\ & + \frac{1}{2} \nu_4^f (\delta_{ik}^{\perp} \delta_{jl}^{\perp} + \delta_{il}^{\perp} \delta_{jk}^{\perp}). \end{aligned} \quad (54)$$

From eqs. (43) and (45) we get the dissipative parts of the phenomenological currents (disregarding  $O(\nabla \mathbf{m}, \nabla \mathbf{h})$ )

$$\begin{aligned} j_i^{\sigma,D} = & -\partial R / \partial \nabla_i T \\ = & -\kappa_{ij} \nabla_j T - \rho \phi (1 - \phi) d_{ij}^T \nabla_j \Pi', \end{aligned} \quad (55)$$

$$\begin{aligned} j_i^{\phi,D} = & -\partial R / \partial \nabla_i \Pi' \\ = & -\rho d_{ij} \nabla_j \Pi' - \rho \phi (1 - \phi) d_{ij}^T \nabla_j T, \end{aligned} \quad (56)$$

$$\sigma_{ij}^D = -\partial R / \partial A_{ij} = -\nu_{ijkl} \nabla_l v_k, \quad (57)$$

$$Y_i^D = \partial R / \partial h_i = b^D \delta_{ij}^{\perp} h_j, \quad (58)$$

$$X^D = \partial R / \partial m = \xi' m, \quad (59)$$

where we have introduced the usual form of the diffusion ( $D_{ij} = \rho d_{ij}$ ) as well as the thermo-diffusion tensor ( $D_{ij}^T = \rho \phi (1 - \phi) d_{ij}^T$ ). The ratios  $d_{ij}^T / d_{*}$  (with  $* \in \{\perp, \parallel\}$ ) and  $d_{*}^T / \kappa_{*}$  are called the Soret and the Dufour coefficients, respectively (the latter being usually neglected in liquids). The last equation describes the relaxation of the amplitude  $F$  of the relative velocity to its stationary value  $F_s$ , which is characteristic for active dynamic processes.

Although the dissipation function given above is bilinear in the forces, the dissipative currents contain nonlinearities due to the (implicit and explicit) dependences of transport tensors on the variables.

## 2.6 Reversible dynamics

The phenomenological reversible currents (superscript  $R$ ) are most easily derived by writing down all symmetry-allowed contributions to the various currents and make then sure that the entropy production is zero

$$R = -j_i^{\sigma,R} \nabla_i T - j_i^{\phi,R} \nabla_i \Pi' - \sigma_{ij}^R A_{ij} + Y_i^R h_i + X^R m = 0. \quad (60)$$

The reversible currents have the same time reversal behavior as the time derivative of the appropriate variable. They are either even under spatial inversion,  $\sigma_{ij}^R, X^R$ , or polar vectors,  $j_i^{\phi,R}, j_i^{\sigma,R}, Y_i^R$ , where the latter is transverse to  $\mathbf{f}$ . Typically, zero-entropy condition is achieved by cancellation of two mutual cross-coupling terms, in particular we find

$$j_i^{\sigma,R} = \beta_{\parallel} f_i m + \beta_{\perp} \delta_{ij}^{\perp} h_j, \quad (61)$$

$$j_i^{\phi,R} = \gamma_{\parallel} f_i m + \gamma_{\perp} \delta_{ij}^{\perp} h_j, \quad (62)$$

$$\sigma_{ij}^R = a_{ij} m + \lambda_{ijk} h_k, \quad (63)$$

$$Y_i^R = \delta_{ij}^{\perp} (\beta_{\perp} \nabla_j T + \gamma_{\perp} \nabla_j \Pi' + \beta_{\parallel} \nabla_j m) + \lambda_{kji} A_{jk} + \beta_3 \delta_{ik}^{\perp} f_j (\nabla_j F_k) m + \beta_4 \delta_{ik}^{\perp} f_j \omega_{kj} m, \quad (64)$$

$$X^R = \beta_{\parallel} f_i \nabla_i T + \gamma_{\parallel} f_i \nabla_i \Pi' + a_{ij} A_{ij} + \beta_1 \delta_{ij}^{\perp} \nabla_j h_i - \beta_3 \delta_{ik}^{\perp} f_j (\nabla_j F_k) h_i + \beta_4 \delta_{ik}^{\perp} f_j \omega_{jk} h_i, \quad (65)$$

with  $a_{ij} = a_{\parallel} f_i f_j + a_{\perp} \delta_{ij}^{\perp}$  and  $\lambda_{ijk} = \lambda (\delta_{ik}^{\perp} f_j + \delta_{jk}^{\perp} f_i)$ . The latter is the flow alignment term describing orientational changes of the preferred direction due to shear flow (and vice versa back flow) as is well known from (passive) nematic liquid crystals. The former couples compressional flow to changes of the degree of activity and vice versa. It has the form  $\sigma_{ij} \sim f_i f_j (F - F_s)$  resembling the so-called active terms in the nematic and static polar description of active systems. There are (reversible) cross-couplings linking the heat and concentration currents to changes of the degree of activity ( $\beta_{\parallel}, \gamma_{\parallel}$ ) and to orientational deformations ( $\beta_{\perp}, \gamma_{\perp}$ ), which in turn lead to temporal changes of the degree of activity and to structure rotations due to temperature and osmotic pressure gradients. These couplings are specific for dynamic polar ordering and occur for dynamic axial structures only in the chiral case.

There are also reversible cross-couplings between the structure and the degree of activity, both linear ( $\beta_1$ ) and nonlinear ( $\beta_3, \beta_4$ ). The latter have been kept in eq. (64), since they are relevant for the question of transport convection velocities.

## 2.7 Chirality and macroscopic hand

Biologically active systems are very often chiral —*i.e.* they are characterized by a pseudoscalar quantity,  $q_0$ , that changes sign under spatial inversion. As a result a mirror image of such a system is not equivalent to the original (structures cannot be aligned by rotations and translations, only). In the bulk this leads frequently to a helical superstructure as a consequence of the collective handedness. However, even in monolayers, which are of high biological relevance as well [42], consequences of a macroscopic hand have been found and analyzed experimentally [42] as well as studied theoretically [43, 44]. Even for isotropic systems containing chiral objects consequences of a macroscopic hand are of interest [45].

In the present context of an active system with a polar dynamic preferred direction the reduced symmetry ( $C_{\infty}$  compared to the achiral  $C_{\infty v}$  case) allows additional terms in the chiral internal energy

$$\epsilon_{\text{chir}} = \epsilon + q_0 \tilde{K}_2 \left( \mathbf{f} \cdot [\nabla \times \mathbf{F}] + \frac{1}{2} q_0 \right) - q_0 (\mathbf{f} \cdot [\nabla \times \mathbf{F}] + q_0) (\tau_{\phi} \delta \phi + \tau_{\sigma} \delta \sigma + \tau_{\rho} \delta \rho), \quad (66)$$

where  $\epsilon$  is the achiral energy given by eq. (21). The two types of contributions represent the linear twist giving rise to helical structures in the bulk, and the static Lehmann-type terms coupling structure rotation with concentration, entropy density and density changes, respectively. The same types of chiral contributions are present in (passive) cholesteric as well as in chiral non-polar dynamic active systems. They also exist for polar cholesterics for which they have never been discussed in the literature so far. However, the splay-twist coupling term,  $(\nabla_i F_i) F_j (\nabla \times \mathbf{F})_j$ , the hallmark of (passive) polar cholesterics [46], is not possible in the present class of active systems, due to time reversal symmetry.

The dissipative dynamics also shows additional chiral contributions

$$R_{\text{chir}} = R + q_0 \epsilon_{ijk} f_k h_j (\psi_{\phi} \nabla_i \Pi' + \psi_{\sigma} \nabla_i T + \zeta_m \nabla_i m) + \zeta_A f_l A_{il} + \zeta_h f_l \nabla_l h_i, \quad (67)$$

where  $R$  is the achiral entropy production given in eq. (45). The chiral terms in the first row are of the standard dynamic Lehmann type [47]<sup>1</sup>, where constant gradients of the scalar variables give rise to rotations of the preferred direction<sup>2</sup>, well-known from (passive) cholesterics as well as from chiral non-polar dynamic active systems. In the former system the analogue to the  $\nabla_i m$  is a gradient of the strength of the nematic order parameter  $S$ . In addition, such terms also exist in polar cholesterics, for which Lehmann-type effects have never been studied in the literature. The second row constitutes dissipative couplings among flow and orientation. The first term in the second row has an achiral analogue for uniaxial magnetic gels [40] as well as for an axial dynamic preferred direction [22]. It describes *e.g.*, for a chiral monolayer with planar orientation of  $\mathbf{f}$ , that an in-plane shear flow leads to a rotation of  $\mathbf{f}$  out of the plane. The last term in the second row is of higher gradient order than the orientational relaxation ( $\sim b^D$ ) in eq. (45), but of lower order than the  $\nu^{ff}$  term in eq. (48).

Finally, reversible couplings between flow and temperature and concentration gradients are possible in the chiral case

<sup>1</sup> From the analysis of ref. [47] it became clear that the original experiments of Otto Lehmann on cholesterics droplets in a temperature gradient [48] revealed a mixture of static and dissipative dynamic cross-coupling effects.

<sup>2</sup> A possible coupling to an electric field,  $\psi_e E_i$ , is not considered here.

$$j_i^{\sigma,R,\text{chir}} = \frac{q_0}{2} g_\sigma f_j (f_l \epsilon_{ijk} + f_k \epsilon_{ijl}) A_{kl}, \quad (68)$$

$$j_i^{\phi,R,\text{chir}} = \frac{q_0}{2} g_\phi f_j (f_l \epsilon_{ijk} + f_k \epsilon_{ijl}) A_{kl}, \quad (69)$$

$$\sigma_{ij}^{R,\text{chir},1} = \frac{q_0}{2} f_l (f_j \epsilon_{ilk} + f_i \epsilon_{jlk}) (g_\sigma \nabla_k T + g_\phi \nabla_k \Pi'), \quad (70)$$

not possible in the achiral currents eqs. (61)-(63). In addition, there is a reversible flow contribution to the stress tensor

$$\sigma_{ij}^{R,\text{chir},2} = q_0 f_p (\epsilon_{ikp} \nu_{jl}^R + \epsilon_{jkp} \nu_{il}^R + \epsilon_{jlp} \nu_{ik}^R + \epsilon_{ilp} \nu_{jk}^R) A_{kl}, \quad (71)$$

with  $\nu_{ij}^R = \nu_\perp^R \delta_{ij}^\perp + \nu_\parallel^R f_i f_j$ . There is an equivalent, albeit achiral, reversible flow coupling in non-polar dynamic active systems [22].

### 3 Simple solutions, the structure of the normal modes, and comparison with other systems

#### 3.1 Transport and convection velocities

The system of dynamic equations has been set up by choosing the mean velocity as transport velocity for all variables. This leads to a thermodynamic valid description, which however might not be the physically relevant one. Assuming the two-fluid system consists of two only weakly interacting subsystems, one would expect all variables belonging to subsystem “1” (“2”) to be advected and convected with velocity  $\mathbf{v}_1$  ( $\mathbf{v}_2$ ). Fortunately, our choice for the transport velocity is not unique, since the phenomenological currents derived above can change the effective transport velocity for some of the variables. *E.g.* the  $\gamma_\parallel$  term on eq. (62) allows to write the concentration dynamics eq. (37) as

$$\dot{\phi} + v_j \nabla_j \phi + \gamma_\parallel (1 - 2\phi) F_j \nabla_j \phi + \dots, \quad (72)$$

where the  $\dots$  contain all contributions not related to the transport of  $\phi$ . For the mass currents of the individual densities, eqs. (1) and (2), this means

$$j_i^{(1)} = \rho_1 v_i^{(1)} + (\gamma_\parallel - 1) \alpha F_i + \dots, \quad (73)$$

$$j_i^{(2)} = \rho_2 v_i^{(2)} - (\gamma_\parallel - 1) \alpha F_i - \dots \quad (74)$$

For  $\gamma_\parallel = 0$  the transport velocity is  $\mathbf{v}$  for both of them, while for  $\gamma_\parallel = 1$  the densities  $\rho_1$  and  $\rho_2$  are advected with velocity  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , respectively. This is indeed a choice that is used in many simple 2-fluid models. For the concentration  $\phi$  the transport velocity in this case is the skew velocity  $\mathbf{v}_{sk} = (1 - \phi) \mathbf{v}_1 + \phi \mathbf{v}_2$ . In real systems with  $0 \leq \gamma_\parallel \leq 1$  the transport velocity is a material property and cannot be fixed *a priori* by general symmetry properties.

The transport velocity of the heat transport can only be tuned by a nonlinear effect: Assuming a temperature

(entropy) dependence of the reversible transport parameter,  $\beta_\parallel = \beta_\parallel^0 + \beta_\parallel^1 \sigma$ , allows to write eq. (36) as

$$\dot{\sigma} + \nabla_j \sigma (v_j + \beta_\parallel^1 F_j) + \dots, \quad (75)$$

leading to an effective transport velocity, *e.g.*  $\mathbf{v}_1$ ,  $\mathbf{v}_{sk}$ , or  $\mathbf{v}_2$ , for  $\rho \beta_\parallel^1 = \rho_2, \rho_2 - \rho_1, -\rho_1$ , respectively. Since the heat transport cannot simply be attributed to a specific subsystem, choosing the mean velocity as the transport velocity ( $\beta_\parallel^1 = 0$ ) might be reasonable, if the actual value of  $\beta_\parallel^1$  is unknown.

Similarly, the transport and convection properties of  $\mathbf{F}$  (or rather  $\mathbf{f}$ ) are modified by the phenomenological reversible currents. Inspection of eq. (64) reveals that in eqs. (39) and (41) the effective transport and convection velocities are  $\mathbf{v} + \beta_3 m \mathbf{f}$  and  $\mathbf{v}(1 - \beta_4 m/F)$ , respectively. These are nonlinear effects, since  $m$  depends on  $F - F_s$  and on the distortions of the polar structure. By choosing  $\beta_3 = (1/\rho_1) - (1/\rho_2)$ , the advection velocity is  $\mathbf{v}_{sk} + (\rho_1 - \rho_2) F_s \mathbf{f}$ . This choice also has implications to the advection of the individual momenta  $\mathbf{g}_1$  and  $\mathbf{g}_2$ . In particular, for a passive system ( $F_s = 0$ ) it means they are advected by  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , respectively [30] as is assumed in many simplified *ad hoc* models. In an active system, however, such a simple advection behavior is not possible. Similarly, for  $\beta_4 = \rho/(\rho_1 \rho_2)$  in a passive system, neither the relative velocity nor the individual momenta  $\mathbf{g}_1$  and  $\mathbf{g}_2$  are convected at all, while in an active one this is not possible.

Generally, in a two-fluid system the advection and convection velocities are material dependent. In an active system, by tuning  $F_s$ , the transport and convection velocities can be varied in an even wider range.

#### 3.2 Normal mode structure and comparison

The normal mode structure of a system reflects its linear dynamic properties. In most cases it is accessible by experiments and allows an easy connection with the theoretical description. For the active polar system there are 9 dynamic variables, which are coupled on the static as well as the dynamic level. It is obvious that the resulting ninth-power algebraic equation, whose solutions are the desired dispersion relations of the modes, cannot be discussed analytically in any straightforward manner. Since the hydrodynamic description is basically a gradient expansion, it is appropriate to discuss the normal mode structure in terms of increasing powers of gradients involved.

There is one spatially homogeneous relaxation mode in the system present, the relaxation of the degree of activity  $F(t) - F_s \sim \exp(-\xi t)$  with  $\xi = \xi' \alpha$ , cf. eqs. (26), (40), and (59). A similar relaxation mode is found in the conventional (passive) two-fluid models (of *e.g.* immiscible binary mixtures), where a friction force proportional to the velocity difference of the two subsystems leads to a homogeneous relaxation of this velocity difference to zero for long times [30]. There is no homogeneous relaxation mode in the two-fluid description of superfluids, since the



broken-symmetry origin of the second (superfluid) velocity leads to an effective conservation behavior and does not allow for a relaxation [31].

In a second step we discuss the inhomogeneous modes in lowest gradient order. In this order, no dissipative effects are present. We assume all variables (or rather their deviations from the stationary values indicated by the subscript “s”), including  $\nabla_i F$ , to have a time and space dependence proportional to  $\exp i(\omega t - \mathbf{k} \cdot \mathbf{r})$ . Linearizing the complete set of equations, non-trivial solutions (with finite amplitudes) are only possible for certain frequency values  $\omega = \omega(k)$ , the dispersion relations. In lowest order  $\omega(k) \sim k$  we are conveniently left with 5 dynamic equations for *e.g.*  $\dot{\rho}$ ,  $\dot{\phi}$ ,  $\dot{\sigma}$ ,  $\text{div } \dot{\mathbf{v}}$ , and  $\nabla_{\parallel} \dot{m}$ , if we assume for the moment the  $a_{ij}$  tensor in eqs. (63) and (65) to be isotropic ( $a_{\perp} = a_{\parallel}$ ). After some algebra we find for four modes

$$\omega^4 - \omega^2 \left( c_{\perp}^2 k^2 + \alpha' Q_2 k_{\parallel}^2 \right) + \omega k^2 k_{\parallel} \alpha' Q_1 + k^2 k_{\parallel}^2 \alpha' Q_0 = 0, \quad (76)$$

where  $\alpha' = \phi_s(1 - \phi_s)$ . The coefficients  $Q_{0,1,2}$  are combinations of products between static susceptibilities (*e.g.* eq. (16)) and the reversible transport parameters  $\gamma_{\parallel}$  and  $\beta_{\parallel}$ , cf. eqs. (61), (62), and (65). The expressions are too bulky to be given here in detail.

Several qualitative conclusions can be drawn from eq. (76). First, the modes are anisotropic. For  $k_{\parallel} = 0$  (where  $\parallel$  refers to the direction parallel to  $f_i$ ) there is first sound  $\omega_{1,2} = \pm c_{\perp} k_{\perp}$ , but no second sound,  $\omega_{3,4} = 0$ . The transverse sound velocity

$$c_{\perp}^2 = c_0^2 - \frac{1}{2} \rho_s F_s^2 (1 - 2\phi_s) + \alpha' a_{\perp}^2 \quad (77)$$

is different from the ordinary isotropic fluid sound velocity with  $c_0^2 = \partial p / \partial \rho|_{\sigma}$  due to the presence of activity. If the wave propagation has a longitudinal component  $k_{\parallel}$ , there is a kind of second sound  $\omega_{3,4} = c_{3,4} k_{\parallel}$ , where one can get an analytical expression for  $c_{3,4}$  in the limit  $k_{\parallel} \ll k_{\perp}$

$$c_{3,4} = \frac{\alpha'}{2c_{\perp}^2} \left( Q_1 \pm \sqrt{Q_1^2 - 4c_{\perp}^2 Q_0} \right). \quad (78)$$

As long as  $Q_1^2 > 4c_{\perp}^2 Q_0$  these are two truly propagating waves, while in the opposite case they are overdamped. Note that  $c_3 \neq -c_4$  indicating the broken time reversal symmetry in an active non-equilibrium state with a finite velocity  $\mathbf{F}_s$ . This is already manifest in eq. (76), where the linear frequency term ( $\sim Q_1$ ) destroys the  $\omega \rightarrow -\omega$  symmetry. There is no second sound in conventional (passive) two-fluid models and in superfluids second sound is a Goldstone mode that is always propagating.

First sound is anisotropic and the longitudinal sound velocity is different from the transverse one,  $c_{\perp}$ , and from the ordinary isotropic sound velocity,  $c_0$ . For  $k_{\perp} = 0$  first sound takes the form  $\omega_{1,2} = c_{1,2} k_{\parallel}$  with

$$c_{1,2} = \pm c_{\perp} \pm \frac{\alpha'}{2c_{\perp}} \left( Q_2 - \frac{Q_0}{c_{\perp}^2} \right) - \alpha' \frac{Q_1}{2c_{\perp}^2}, \quad (79)$$

if one assumes that  $c_{\perp}$  is the dominating contribution. The broken time reversal symmetry due to the active velocity  $\mathbf{F}_s$  is reflected by  $c_1 \neq -c_2$ .

We note that first sound is also becoming anisotropic in superfluid  $^3\text{He-A}$ , when the phase transition to the isotropic phase is approached from below and the modulus of the order parameter is added to the hydrodynamic description as a macroscopic variable [49].

There is a fifth, but trivial, mode  $\omega_5 = 0$ , since  $\dot{\rho}/\rho_s - \dot{\sigma}/\sigma_s - (\beta_{\parallel} \rho_s / \gamma_{\parallel} \sigma_s) \dot{\phi} \equiv 0$  in order  $\omega \sim k$ . Relaxing the condition  $a_a = a_{\parallel} - a_{\perp} = 0$  introduces a sixth dynamic equation (for  $\nabla_{\parallel} \dot{v}_{\parallel}$ ), but does not change any of the qualitative results given above. Only another trivial mode  $\omega_6 = 0$  (due to  $\nabla_{\parallel} \text{div } \dot{\mathbf{v}} - \Delta \dot{v}_{\parallel} + (a_a / \gamma_{\parallel}) \Delta \dot{\phi} \equiv 0$ ) is introduced and the coefficient functions  $Q_{0,1,2}$  become even more involved.

Other, transverse variables, like  $f_i$  do not contribute to the sound spectrum in order  $\omega \sim k$ , but are affected by (first) sound excitations via  $\dot{\mathbf{f}} \sim \mathbf{k}_{\perp} (\beta_{\perp} \delta T + \gamma_{\perp} \delta \Pi' + \beta_{\perp} \delta m)$  according to eq. (64). This feature is unknown in ordinary nematics, while shear alignment, the orientation of the preferred direction  $f_i$  in simple shear flow, is the same as in nematics.

In order  $\omega \sim k^2$  the dissipative effects come into play, generally leading to complex contributions to the mode frequencies of the form  $\sim ik^2$ . However, on this level all variables are coupled and it is impossible to give an explicit analytical discussion of the modes.

### 3.3 Flow induced heat current

Here we will discuss experimental consequences of a reversible coupling term between symmetrized velocity gradients,  $A_{ij}$  and the heat current, which is allowed for a chiral system with a polar dynamic preferred direction. From eq. (68) we obtain assuming  $\mathbf{f} \parallel \hat{\mathbf{z}}$

$$j_i^{\sigma,R,\text{chir}} = q_0 g_{\sigma} \epsilon_{izk} A_{kz}. \quad (80)$$

One immediately realizes that only for shear or elongational flows applied in a plane containing the polar dynamic preferred direction a heat current arises. And this heat current is perpendicular to the flow plane. For example, for a shear flow in the  $(y, z)$ -plane,  $\nabla_k v_j = 2S \delta_{ky} \delta_{jz}$  with the shear rate  $S$ , we find

$$j_x^{\sigma,R,\text{chir}} = -q_0 g_{\sigma} S. \quad (81)$$

There is the opposite effect that a temperature gradient perpendicular to the preferred direction ( $\nabla_i T = 2\Gamma \delta_{ix}$ ) introduces (elongational) flow in the plane perpendicular to it,  $\sigma_{yz} = \sigma_{zy} = q_0 g_{\sigma} \Gamma$ . We note that a macroscopic hand or collective chirality, as indicated by the presence of the factor  $q_0$ , is a prerequisite for these effects to occur. For non-chiral systems such effects are not allowed by symmetry. The effects are reversible meaning the reversal of the shear velocity will result in a reversal of the heat current, and vice versa. We would also like to emphasize that this analysis is not restricted to the coupling of an

extensional flow to the heat current, but is expected to arise equally well for concentration currents or for electric currents in case the effect of electric fields is taken into account.

It is instructive to compare the scenario just discussed with a similar one that can be found in tetrahedric systems, which show octupolar order [50] and thus break parity symmetry in their ground state. For such (achiral) tetrahedric systems it has been shown [51, 52] that there are also reversible dynamic contributions coupling  $A_{ij}$  to the entropy current etc., which take the form [51]

$$j_i^{\sigma,R,\text{oct}} = \Gamma_2 T_{ijk} A_{jk}, \quad (82)$$

with  $T_{ijk}$  the tetrahedric order parameter. A simple shear flow applied in the plane perpendicular to one of the tetrahedral directions leads to a heat current parallel to this tetrahedral direction, *e.g.* applying the flow in the  $(y, z)$ -plane, that is  $A_{jk} = S\delta_{jy}\delta_{kz}$  with the shear rate  $S$ , we find [51], making use of Fel's normalization of  $T_{ijk}$  [50]

$$j_x^{\sigma,R,\text{oct}} = \Gamma_2 \frac{4S}{3\sqrt{3}}. \quad (83)$$

Thus the picture emerging for tetrahedratics is quite complementary to that found for a chiral system with a dynamic polar preferred direction. While in the latter case a shear flow applied in the plane containing the polar dynamic preferred direction leads to a heat current perpendicular to this plane, for tetrahedric systems the heat current is parallel to the chosen tetrahedric direction and perpendicular to the shear flow.

Furthermore it appears important to stress that for systems with a polar dynamic preferred direction the reversible coupling discussed only exists for a system with a macroscopic hand. Without chirality such a coupling is ruled out by symmetry. This observation could pave the way for finding clear-cut experimental evidence for the role played by chirality as a collective effect in biological systems.

## 4 Conclusions and perspective

By introducing a (second) active velocity we have developed in this manuscript a macroscopic description of active systems with dynamic polar order. This approach is conceptually different from descriptions that use static (polar) nematic order parameters (see [26, 27] and quite recently [53]). As a result, we find a normal mode structure quite different compared to the static descriptions, as well as linear couplings between (active) flow and *e.g.* densities and concentrations due to the genuine two-fluid transport derivatives. On the other hand, we get, quite similar to the static case, a direct linear relation between the stress tensor and the structure tensor. This prominent "active" term is responsible for many active effects, meaning that our approach can describe those effects as well. In addition, we also deal with explicitly chiral systems, which are important for many active systems. In particular, we

find an active flow-induced heat current specific for the dynamic chiral polar order.

The present general description of active dynamic polar order is expected to be applicable to systems, where two different velocities, for the active and the passive part, are important. Among those one can think of bird flocks or insect (locust) swarms flying in windy air, fish schools moving in water currents, or bio-convection of bacteria colonies in a solvent background. In our general treatment of dynamic polar order we have taken the active velocity as a given quantity, without specifying how it arises from the physical, chemical or biological driving forces. The latter are highly system specific and require additional (often heuristic) modeling, which goes beyond the scope of the present manuscript. Such a model for the stop-and-go dynamics of growing *Proteus mirabilis* bacteria, based on the active velocity as order parameter, has been developed recently [29].

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